# Stratifications of Structured Matrix Pencils
# **Part II: Generalized State-Space Systems**

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# Non-singular generalized state-space system

We study the *non-singular generalized state-space system*

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \qquad \det E \neq 0 \\ y(t) &= Cx(t) + Du(t), \end{aligned}$$

where $E, A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times m}$, $C \in \mathbb{C}^{p\times n}$, $D \in \mathbb{C}^{p\times m}$, and $x(t), y(t)$, and $u(t)$ are *state, output,* and *input* vectors

# System pencil

The associated *system pencil* to a non-singular generalized state-space system is

$$\mathcal{S} := \begin{bmatrix} A & B \\ C & D \end{bmatrix} - \lambda \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}, \qquad \det E \neq 0$$

$\widetilde{\mathcal{S}}$ is *feedback-injection equivalent* to $\mathcal{S}$ if and only if

$$\begin{bmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{bmatrix} = \begin{bmatrix} R_1 & R_2 \\ 0 & R_3 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} S_1 & 0 \\ S_2 & S_3 \end{bmatrix},$$

$$\begin{bmatrix} \widetilde{E} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} R_1 & R_2 \\ 0 & R_3 \end{bmatrix} \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} S_1 & 0 \\ S_2 & S_3 \end{bmatrix}, \qquad (\widetilde{E} = R_1 E S_1),$$

where $\det R_1 \cdot \det R_3 \cdot \det S_1 \cdot \det S_3 \neq 0$

Orbit: $\mathcal{O}_{\text{F-I}}(\mathcal{S}) = \{\widetilde{\mathcal{S}}$ that are feedback-injection equivalent to $\mathcal{S}\}$

# Block direct sum

By the *block direct sum* of

$$\mathcal{S} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} - \lambda \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathcal{S}' = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} - \lambda \begin{bmatrix} E' & 0 \\ 0 & 0 \end{bmatrix},$$

we mean

$$\mathcal{S} \boxplus \mathcal{S}' := \begin{bmatrix} A \oplus A' & B \oplus B' \\ C \oplus C' & D \oplus D' \end{bmatrix} - \lambda \begin{bmatrix} E \oplus E' & 0 \\ 0 & 0 \end{bmatrix}$$

# Blocking of system pencils

To define a blocking of

$$\mathcal{S} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} - \lambda \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}$$

we write $\mathcal{S}_{pm}$, where $p \times m$ is the size of $D$

For example

$$\mathcal{S}_{11} := \left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 \end{array}\right] - \lambda \left[\begin{array}{cccc|c} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

$$\mathcal{S}_{10} := \left[\begin{array}{ccc} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] - \lambda \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 0 & 0 & 0 \end{array}\right]$$

# Canonical building blocks

$$
J_i(\mu) - \lambda I_i := \begin{bmatrix} \mu & 1 & & \\ & \mu & \ddots & \\ & & \ddots & 1 \\ & & & \mu \end{bmatrix} - \lambda \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}, \qquad i \times i \;\text{ for } i \geq 1,
$$

$$
N_i - \lambda N_i' := \begin{bmatrix} & & & 1 \\ & & \cdot^{\cdot^{\cdot}} & \\ & 1 & & \\ 1 & & & \end{bmatrix} - \lambda \begin{bmatrix} & & 1 & 0 \\ & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \\ 1 & 0 & & \\ 0 & & & \end{bmatrix}, \qquad i \times i \;\text{ for } i \geq 1,
$$

$$
G_j - \lambda H_j := \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}, \qquad j \times (j+1) \;\text{ for } j \geq 0
$$

# Canonical forms

## Theorem 1 (Thorp '73)

Each system pencil $\mathcal{S}$ can be reduced by feedback-injection equivalence transformation to a block direct sum of pencils of the forms

$$(J_i(\mu) - \lambda I_i)_{00},\ (N_i - \lambda N_i')_{11},\ (G_j - \lambda H_j)_{01},\ (G_j^T - \lambda H_j^T)_{10},$$

where $i = 1, 2, \ldots,$ and $j = 0, 1, 2, \ldots$

This sum is uniquely determined up to permutation of summands.

# Orbit stratifications

- Given a system pencil and its orbit:
  What other structures are found within its closure?

**Stratification**

The closure hierarchy of all possible system pencil orbits

# Orbit stratifications

- Given a system pencil and its orbit:
  What other structures are found within its closure?

*Stratification*

The closure hierarchy of all possible system pencil orbits

We use:

- Graphs to illustrate stratifications
- Dominance orderings for integer partitions in proofs and derivations

See also next talk by Andrii!

# Orbit stratifications – some previous results

- Matrix pencils under strict equivalence

$$\mathcal{O}_{\mathrm{E}}(A-\lambda E) = \{P^{-1}(A-\lambda E)Q : \det P \cdot \det Q \neq 0\}$$

A. Edelman, E. Elmroth, B. Kågström, A geometric approach to perturbation theory of matrices and matrix pencils. Part I: Versal deformations, *SIAM J. Matrix Anal. Appl.* 18(3):653–692 1997.
**SIAM/SIAG Linear Algebra Prize 2000**

A. Edelman, E. Elmroth, B. Kågström, A geometric approach to perturbation theory of matrices and matrix pencils. Part II: A stratification-enhanced staircase algorithm. *SIAM J. Matrix Anal. Appl.*, 20(3):667–699, 1999.

- Controllability and observability pairs under feedback equivalence

$$\mathcal{O}_{\mathrm{FE}}([A-\lambda E\ B]) = \left\{P^{-1}\,[A-\lambda E\ B]\begin{bmatrix} P & 0 \\ R & Q \end{bmatrix} : \det T \cdot \det P \cdot \det Q \neq 0\right.$$

$$\mathcal{O}_{\mathrm{FE}}\left(\begin{bmatrix} A-\lambda E \\ C \end{bmatrix}\right) = \left\{\begin{bmatrix} P^{-1} & R \\ 0 & Q \end{bmatrix}\begin{bmatrix} A-\lambda E \\ C \end{bmatrix} P : \det T \cdot \det P \cdot \det Q \neq 0\right\}$$

E. Elmroth, S. Johansson, B. Kågström, Stratification of controllability and observability pairs – theory and use in applications. *SIAM J. Matrix Anal. Appl.* 31(2):203–226 2009.

# Comparison of the canonical forms

| Investigated pencils | Possible canonical summands |
|---|---|
| $\begin{bmatrix} A & B \end{bmatrix} - \lambda \begin{bmatrix} E & 0 \end{bmatrix}$, det $E \neq 0$, | $(J_i(\mu) - \lambda I_i)_{00}$, $(G_j - \lambda H_j)_{01}$ |
| $\begin{bmatrix} A \\ C \end{bmatrix} - \lambda \begin{bmatrix} E \\ 0 \end{bmatrix}$, det $E \neq 0$ | $(J_i(\mu) - \lambda I_i)_{00}$, $(G_j^T - \lambda H_j^T)_{10}$ |
| $\begin{bmatrix} A & B \\ C & D \end{bmatrix} - \lambda \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}$, det $E \neq 0$ | $(J_i(\mu) - \lambda I_i)_{00}$, $(G_j - \lambda H_j)_{01}$, $(G_j^T - \lambda H_j^T)_{10}$, $(N_i - \lambda N_i')_{11}$ |
| $A - \lambda E$ | $J_i(\mu) - \lambda I_i$, $G_j - \lambda H_j$, $G_j^T - \lambda H_j^T$, $I_i - \lambda J_i(0)$ |

# The Kronecker canonical form of system pencils

Any system pencil $\mathcal{S}$ can be written as

$$\boxplus_\alpha \boxplus_i (J_{h_i}(\mu_\alpha) - \lambda I_{h_i})_{00} \boxplus \boxplus_j (N_{q_j} - \lambda N'_{q_j})_{11} \boxplus \boxplus_s (G_{r_s} - \lambda H_{r_s})_{01} \boxplus \boxplus_t (G_{l_t}^T - \lambda H_{l_t}^T)_{10}$$

To obtain the *Kronecker canonical form* (KCF) of $\mathcal{S}$ we need "to substitute"

$$\begin{array}{ccc} (J_i(\mu) - \lambda I_i)_{00} & \longrightarrow & J_i(\mu) - \lambda I_i \\ (N_i - \lambda N'_i)_{11} & \longrightarrow & I_i - \lambda J_i(0) \\ (G_j - \lambda H_j)_{01} & \longrightarrow & G_j - \lambda H_j \\ (G_j^T - \lambda H_j^T)_{10} & \longrightarrow & G_j^T - \lambda H_j^T \\ \boxplus & \longrightarrow & \oplus \end{array}$$

Thus we get

$$\oplus_\alpha \oplus_i (J_{h_i}(\mu_\alpha) - \lambda I_{h_i}) \oplus \oplus_j (I_{q_j} - \lambda J_{q_j}(0)) \oplus \oplus_s (G_{r_s} - \lambda H_{r_s}) \oplus \oplus_t (G_{l_t}^T - \lambda H_{l_t}^T)$$

# KCF of system pencils

## Theorem 2

There exists a system pencil $\mathcal{S}$ with the set of Kronecker invariants $\{l_i\}$ , $\{r_i\}$ , $\{q_i\}$ , and $\{h_i^\alpha\}$ if and only if

$$\begin{aligned} \sum_i l_i + \sum_i r_i + \sum_\alpha \sum_i h_i^\alpha + \sum_i (q_i - 1) &= n, \\ \sum_i \#\{l_i\} + \sum_i \#\{q_i\} &= m, \\ \sum_i \#\{r_i\} + \sum_i \#\{q_i\} &= p, \end{aligned}$$

where $\#\{x_i\}$ denotes the number of elements in the set $\{x_i\}$.

# Versal deformations

$$\mathcal{S} := \begin{bmatrix} A & B \\ C & D \end{bmatrix} - \lambda \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}, \qquad \mathcal{W} = \begin{bmatrix} W_1 & W_2 \\ W_3 & W_4 \end{bmatrix} - \lambda \begin{bmatrix} W_5 & W_6 \\ W_7 & W_8 \end{bmatrix},$$

A deformation of a matrix pencil is *versal* if and only if it captures all possible Kronecker Canonical form behaviors near the matrix pencil.

## Lemma 3

Versal deformation of a matrix pencil $\mathcal{S}$ (under strict equivalence) is

$$\mathcal{S} + \mathcal{D}(\mathcal{W}) := \left( \begin{bmatrix} A & B \\ C & D \end{bmatrix} + \begin{bmatrix} W_1 & W_3 \\ W_2 & W_4 \end{bmatrix} \right) - \lambda \left( \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} W_5 & 0 \\ 0 & W_8 \end{bmatrix} \right)$$

where $W_i, i \in \{1, 2, 3, 4, 5, 8\}$ are matrices with arbitrarily small entries (all independent from each other).

# Stratification of system pencils

## Theorem 4

Let $\mathcal{S}$ and $\mathcal{Q}$ be two system pencils. Then there exist $\mathcal{W}$, $U$ and $V$ such that

$$\begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \left( \left( \begin{bmatrix} A & B \\ C & D \end{bmatrix} + \begin{bmatrix} W_1 & W_3 \\ W_2 & W_4 \end{bmatrix} \right) - \lambda \left( \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} W_5 & W_6 \\ W_7 & W_8 \end{bmatrix} \right) \right) \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} = \mathcal{Q}$$

if and only if there exist $\mathcal{W}'$, $U'$ and $V'$ such that

$$\begin{bmatrix} U'_{11} & U'_{12} \\ 0 & U'_{22} \end{bmatrix} \left( \left( \begin{bmatrix} A & B \\ C & D \end{bmatrix} + \begin{bmatrix} W'_1 & W'_3 \\ W'_2 & W'_4 \end{bmatrix} \right) - \lambda \left( \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} W'_5 & 0 \\ 0 & 0 \end{bmatrix} \right) \right) \begin{bmatrix} V'_{11} & 0 \\ V'_{21} & V'_{22} \end{bmatrix} = \mathcal{Q}$$

# Proof of Theorem 4: Necessity

$$\begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \left( \begin{bmatrix} A & B \\ C & D \end{bmatrix} + \begin{bmatrix} W_1 & W_3 \\ W_2 & W_4 \end{bmatrix} - \lambda \left( \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} W_5 & W_6 \\ W_7 & W_8 \end{bmatrix} \right) \right) \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} = \mathcal{Q}$$

By miniversal deformations

$$\mathcal{S} + \mathcal{D}(\mathcal{W}) := \left( \begin{bmatrix} A & B \\ C & D \end{bmatrix} + \begin{bmatrix} W_1' & W_3' \\ W_2' & W_4' \end{bmatrix} \right) - \lambda \left( \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} W_5' & 0 \\ 0 & W_8' \end{bmatrix} \right)$$

**Lemma 5 (Horn & Johnson, Matrix analysis, '85)**

Let $X \in \mathbb{C}^{m \times n}$, $P \in \mathbb{C}^{m \times m}$, $rank(P) = m$, and $Q \in \mathbb{C}^{n \times n}$, $rank(Q) = n$ then $rank(X) = rank(PXQ)$.

$$\mathcal{S} + \mathcal{W}' := \left( \begin{bmatrix} A & B \\ C & D \end{bmatrix} + \begin{bmatrix} W_1' & W_3' \\ W_2' & W_4' \end{bmatrix} \right) - \lambda \left( \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} W_5' & 0 \\ 0 & 0 \end{bmatrix} \right)$$

# Proof of Theorem 4: Necessity

$$\begin{bmatrix} U'_{11} & U'_{12} \\ U'_{21} & U'_{22} \end{bmatrix} \left( \left[ \begin{matrix} A & B \\ C & D \end{matrix} \right] + \begin{bmatrix} W'_1 & W'_3 \\ W'_2 & W'_4 \end{bmatrix} - \lambda \left( \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} W'_5 & 0 \\ 0 & 0 \end{bmatrix} \right) \right) \begin{bmatrix} V'_{11} & V'_{12} \\ V'_{21} & V'_{22} \end{bmatrix} = \mathcal{Q}$$

**Lemma 6 (de Hoyos '90, García-Planas & Magret '01)**

Two system pencils with non-singular $E$, are feedback-injection equivalent *if and only if* they are strictly equivalent.

Therefore $U'_{21} = 0$ and $V'_{12} = 0$

# Stratification of system pencils

## Theorem 4 (Reformulated)

Let $\mathcal{S}$ and $\mathcal{Q}$ be two system pencils. There exists a sequence of non-singular matrices

$$\left\{ U^{(k)} = \begin{bmatrix} U_{11}^{(k)} & U_{12}^{(k)} \\ U_{21}^{(k)} & U_{22}^{(k)} \end{bmatrix}, \; V^{(k)} = \begin{bmatrix} V_{11}^{(k)} & V_{12}^{(k)} \\ V_{21}^{(k)} & V_{22}^{(k)} \end{bmatrix} \right\}, \quad \text{such that}$$

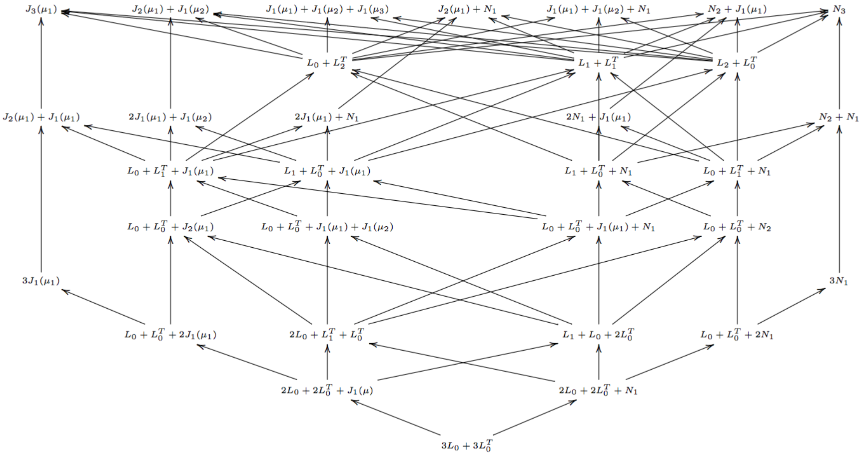
$$U^{(k)} \mathcal{Q} V^{(k)} \to \mathcal{S}$$

if and only if there exists a sequence of non-singular matrices

$$\left\{ U'^{(k)} = \begin{bmatrix} U_{11}'^{(k)} & U_{12}'^{(k)} \\ 0 & U_{22}'^{(k)} \end{bmatrix}, \; V'^{(k)} = \begin{bmatrix} V_{11}'^{(k)} & 0 \\ V_{21}'^{(k)} & V_{22}'^{(k)} \end{bmatrix} \right\}, \text{ such that}$$

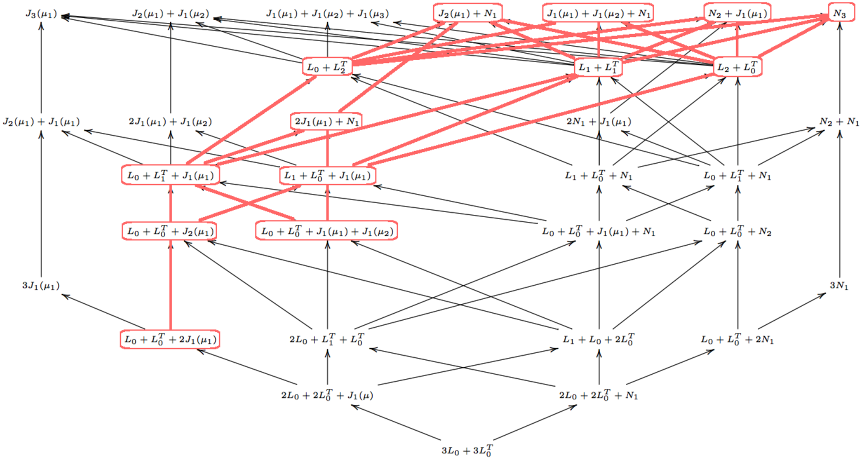
$$U'^{(k)} \mathcal{Q} V'^{(k)} \to \mathcal{S}.$$

# Orbit stratification of 3 × 3 matrix pencils




Remark: Finite and infinite eigenvalues are handled separately

# Orbit stratification of $(2+1) \times (2+1)$ system pencils




Rules for obtaining all the closest neighbours
(cover relations) in the hierarchy are derived too

As done in our earlier work!

# Weyr characteristics

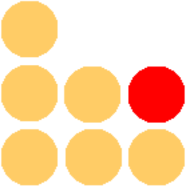
With $\mathcal{S}$ we associate the set of structure integer partitions $\mathcal{R}(\mathcal{S}), \mathcal{L}(\mathcal{S}), \mathcal{N}(\mathcal{S})$, and $\{\mathcal{J}_{\mu_i}(\mathcal{S}) : i = 1, \ldots, d\}$:

- For each distinct $\mu_i$ we have $\mathcal{J}_{\mu_i}(\mathcal{S}) = (h_1^{\mu_i}, h_2^{\mu_i}, \ldots) : h_k^{\mu_i}$ is the number of Jordan blocks of size greater than or equal to $k$
- $\mathcal{N}(\mathcal{S}) = (n_1, n_2, \ldots) : n_k$ is the number of $N - \lambda N'$ blocks of size greater than or equal to $k$
- $\mathcal{R}(\mathcal{S}) = (r_0, r_1, \ldots)$: $r_k$ is the number of $G - \lambda H$ blocks of size greater than or equal to $k \times (k + 1)$
- $\mathcal{L}(\mathcal{S}) = (l_0, l_1, \ldots))$: $l_k$ is the number of $G^T - \lambda H^T$ of size greater than or equal to $(k + 1) \times k$

# Minimal coin moves

$$\mathcal{S}_1 = J_3(\mu) \boxplus J_3(\mu) \boxplus J_1(\mu) \boxplus N_1 \qquad \mathcal{S}_2 = J_4(\mu) \boxplus J_2(\mu) \boxplus J_1(\mu) \boxplus N_1$$

$\mathcal{J}_\mu(\mathcal{S}_1) = (3, 2, 2)$ :

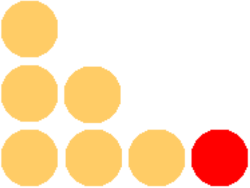


$\mathcal{N}(\mathcal{S}_1) = (1)$ :



$\Longrightarrow$

$\mathcal{J}_\mu(\mathcal{S}_2) = (3, 2, 1, 1)$ :

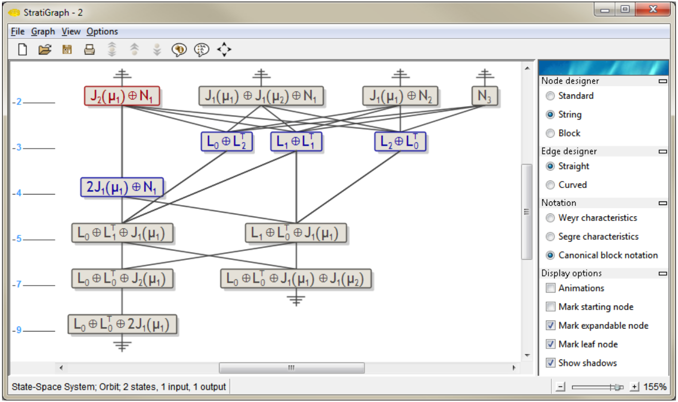


$\mathcal{N}(\mathcal{S}_2) = (1)$ :



$\Rightarrow$ Dominance ordering of integer partitions

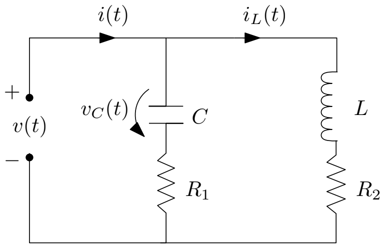
# Orbit stratification of $(2+1)\times(2+1)$ system pencils




StratiGraph: Java Software tool for computing and presenting the stratifications of orbits

B. Kågström, S. Johansson, P. Johansson, *StratiGraph Tool: Matrix Stratifications in Control Applications*, In L. Biegler, S. L. Campbell, and V. Mehrmann (Eds.): Control and Optimization with Differential-Algebraic Constraints, ch. 5, pp. 79–103, SIAM Publications, 2012.

# A $(2+1) \times (2+1)$ example




$$\frac{d}{dt}\begin{bmatrix} v_C \\ i_L \end{bmatrix} = \begin{bmatrix} -\frac{1}{R_1 C} & 0 \\ 0 & -\frac{R_2}{L} \end{bmatrix}\begin{bmatrix} v_C \\ i_L \end{bmatrix} + \begin{bmatrix} \frac{1}{R_1 C} \\ \frac{1}{L} \end{bmatrix} v$$

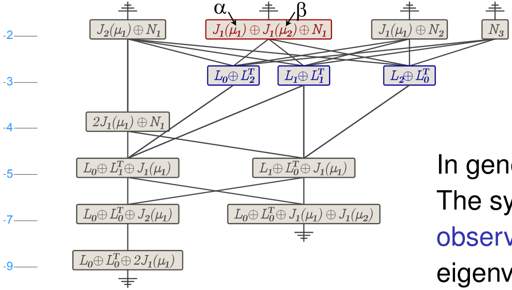
$$i = \begin{bmatrix} -\frac{1}{R_1} & 1 \end{bmatrix}\begin{bmatrix} v_C \\ i_L \end{bmatrix} + \frac{1}{R_1} v$$

with corresponding system pencil

$$\mathcal{S} = \begin{bmatrix} -\frac{1}{R_1 C} - \lambda & 0 & \frac{1}{R_1 C} \\ 0 & -\frac{R_2}{L} - \lambda & \frac{1}{L} \\ -\frac{1}{R_1} & 1 & \frac{1}{R_1} \end{bmatrix}$$

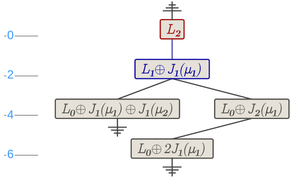
Example from *Linear Systems* by P.J. Antsaklis and A.N. Michel, 2006

# Investigation through stratification

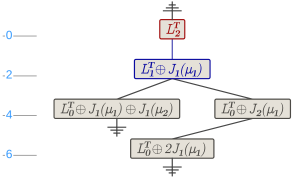



$$\mathcal{S} = \begin{bmatrix} -\frac{1}{R_1 C} - \lambda & 0 & \frac{1}{R_1 C} \\ 0 & -\frac{R_2}{L} - \lambda & \frac{1}{L} \\ -\frac{1}{R_1} & 1 & \frac{1}{R_1} \end{bmatrix}$$

In general:
The system is controllable and observable, where $\mathcal{S}$ has two different eigenvalues $\alpha$ and $\beta$ (transmission zeros of the system $\{A, B, C, D\}$)

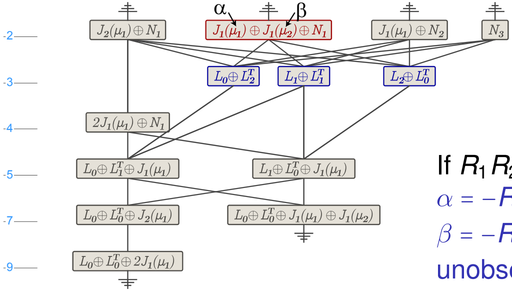



Controllability pair $\mathcal{S}_C = \begin{bmatrix} A - \lambda I & B \end{bmatrix}$



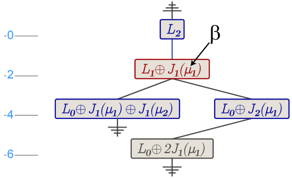

Observability pair $\mathcal{S}_O = \begin{bmatrix} A - \lambda I \\ C \end{bmatrix}$

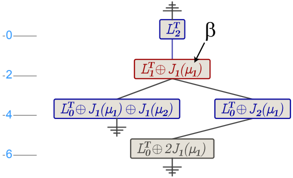
# Investigation through stratification




$$\mathcal{S} = \begin{bmatrix} -\frac{1}{R_1 C} - \lambda & 0 & \frac{1}{R_1 C} \\ 0 & -\frac{R_2}{L} - \lambda & \frac{1}{L} \\ -\frac{1}{R_1} & 1 & \frac{1}{R_1} \end{bmatrix}$$

If $R_1 R_2 C = L$ and $R_1 \neq R_2$:
$\alpha = -R_1/L$ is controllable and observable;
$\beta = -R_2/L$ is uncontrollable and unobservable ($\beta$ is an input-output decoupling zero)

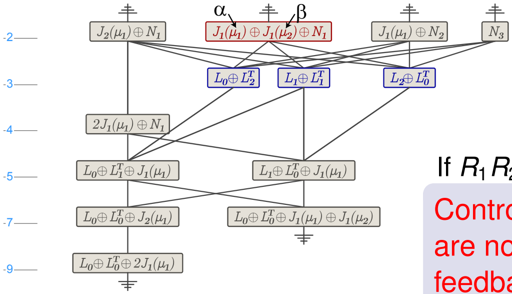



Controllability pair $\mathcal{S}_C = \begin{bmatrix} A - \lambda I & B \end{bmatrix}$

Observability pair $\mathcal{S}_O = \begin{bmatrix} A - \lambda I \\ C \end{bmatrix}$

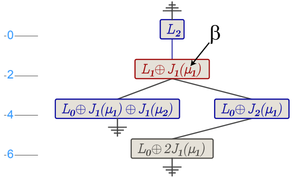
# Investigation through stratification

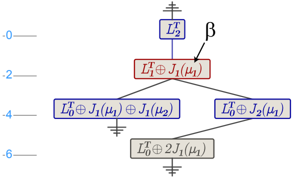



$$\mathcal{S} = \begin{bmatrix} -\frac{1}{R_1 C} - \lambda & 0 & \frac{1}{R_1 C} \\ 0 & -\frac{R_2}{L} - \lambda & \frac{1}{L} \\ -\frac{1}{R_1} & 1 & \frac{1}{R_1} \end{bmatrix}$$

If $R_1 R_2 C = L$ and $R_1 \neq R_2$:

**Controllability and observability are not invariants under feedback-injection equivalence!**

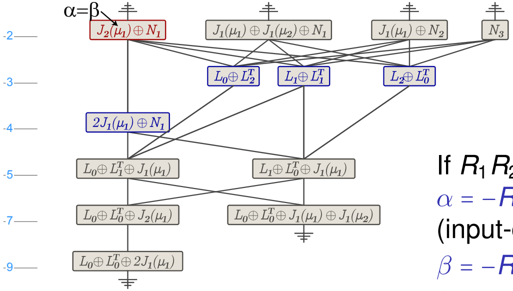



Controllability pair $\mathcal{S}_C = \begin{bmatrix} A - \lambda I & B \end{bmatrix}$

Observability pair $\mathcal{S}_O = \begin{bmatrix} A - \lambda I \\ C \end{bmatrix}$

# Investigation through stratification

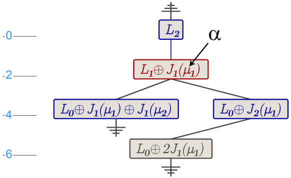



$$\mathcal{S} = \begin{bmatrix} -\frac{1}{R_1 C} - \lambda & 0 & \frac{1}{R_1 C} \\ 0 & -\frac{R_2}{L} - \lambda & \frac{1}{L} \\ -\frac{1}{R_1} & 1 & \frac{1}{R_1} \end{bmatrix}$$

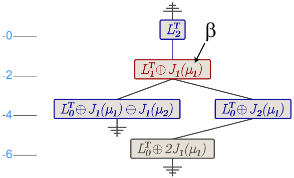
If $R_1 R_2 C = L$ and $R = R_1 = R_2$:

$\alpha = -R/L$ is observable but uncontrollable (input-decoupling zero);

$\beta = -R/L$ is controllable but unobservable (output-decoupling zero)




Controllability pair $\mathcal{S}_C = \begin{bmatrix} A - \lambda I & B \end{bmatrix}$




Observability pair $\mathcal{S}_O = \begin{bmatrix} A - \lambda I \\ C \end{bmatrix}$