Stratifications of structured matrix pencils.
Part I: Skew-symmetric and symmetric cases

Andrii Dmytryshyn[*], Stefan Johansson[*], Bo Kågström[*],
and Vladimir V. Sergeichuk[‡]

[*] Department of Computing Science, Umeå University, Sweden
{andrii, stefanj, bokg}@cs.umu.se
[‡] Institute of Mathematics, Kiev, Ukraine
sergeich@imath.kiev.ua

19^{th} ILAS Conference
August 6–9, 2014, Seoul, Korea

THE e-SCIENCE COLLABORATION
VETENSKAPSRÅDET
THE SWEDISH RESEARCH COUNCIL
UMIT Research Lab
UMEÅ UNIVERSITET

Systems of matrix equations

We study

$$X^T A + AX = 0,$$
$$X^T B + BX = 0,$$

where X is an unknown $n \times n$ matrix and

1. A and B are skew-symmetric $n \times n$ matrices:
   $A, B \in \mathbb{C}^{n \times n}, \quad A = -A^T$ and $B = -B^T$.
2. A and B are symmetric $n \times n$ matrices:
   $A, B \in \mathbb{C}^{n \times n}, \quad A = A^T$ and $B = B^T$.

Results obtained in this part

- The general solutions of

$$\begin{aligned} X^T A + AX &= 0, \\ X^T B + BX &= 0. \end{aligned} \tag{1}$$

- Dimensions of the vector spaces of matrices X that satisfy the systems (1).
- Codimensions of the congruence orbits of a (skew-)symmetric matrix pencil $A - \lambda B$.

Reduction to canonical forms

Multiplying the equations (1) by S^T and S, we obtain

$$S^T X^T S^{-T} \cdot S^T A S + S^T A S \cdot S^{-1} X S = 0,$$
$$S^T X^T S^{-T} \cdot S^T B S + S^T B S \cdot S^{-1} X S = 0,$$

and so the system (1) is equivalent to the system

$$Y^T A' + A' Y = 0,$$
$$Y^T B' + B' Y = 0,$$

where $Y := S^{-1} X S, A' = S^T A S$, and $B' = S^T B S$.

Reduction to canonical forms

Multiplying the equations (1) by S^T and S, we obtain

$$S^T X^T S^{-T} \cdot S^T A S + S^T A S \cdot S^{-1} X S = 0,$$
$$S^T X^T S^{-T} \cdot S^T B S + S^T B S \cdot S^{-1} X S = 0,$$

and so the system (1) is equivalent to the system

$$Y^T A' + A' Y = 0,$$
$$Y^T B' + B' Y = 0,$$

where $Y := S^{-1} X S, A' = S^T A S$, and $B' = S^T B S$.

Sufficient to solve the system (1) where (A, B) is a canonical pair of (skew-)symmetric matrices up to congruence.

Canonical building blocks

Define the $k \times k$ matrices

$$J_k(\mu) := \begin{bmatrix} \mu & 1 & & 0 \\ & \mu & \ddots & \\ & & \ddots & 1 \\ 0 & & & \mu \end{bmatrix}, \qquad I_k := \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}, \quad k = 1, 2, 3, \ldots$$

and the $k \times (k+1)$ matrices

$$F_k := \begin{bmatrix} 1 & 0 & & 0 \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{bmatrix}, \qquad G_k := \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & 1 \end{bmatrix}, \quad k = 0, 1, 2, \ldots$$

Canonical form of a pair (A, B) of skew-symmetric complex matrices under congruence

Theorem (Thompson, 1991)

Every pair of skew-symmetric complex matrices is congruent to a direct sum, determined uniquely up to permutation of summands, of skew-symmetric pairs of the form

$$H_h(\mu) := \left(\begin{bmatrix} 0 & I_h \\ -I_h & 0 \end{bmatrix}, \begin{bmatrix} 0 & J_h(\mu) \\ -J_h(\mu)^T & 0 \end{bmatrix} \right), \quad \mu \in \mathbb{C},$$

$$K_k := \left(\begin{bmatrix} 0 & J_k(0) \\ -J_k(0)^T & 0 \end{bmatrix}, \begin{bmatrix} 0 & I_k \\ -I_k & 0 \end{bmatrix} \right),$$

$$M_m := \left(\begin{bmatrix} 0 & F_m \\ -F_m^T & 0 \end{bmatrix}, \begin{bmatrix} 0 & G_m \\ -G_m^T & 0 \end{bmatrix} \right).$$

The form of the solution

Let (A, B) be a pair of skew-symmetric matrices in canonical form:

$$(A, B) = \bigoplus_{i=1}^{a} H_{h_i}(\mu_i) \oplus \bigoplus_{j=1}^{b} K_{k_j} \oplus \bigoplus_{r=1}^{c} M_{m_r}.$$

The form of the solution

Let (A, B) be a pair of skew-symmetric matrices in canonical form:

$$(A, B) = \bigoplus_{i=1}^{a} H_{h_i}(\mu_i) \oplus \bigoplus_{j=1}^{b} K_{k_j} \oplus \bigoplus_{r=1}^{c} M_{m_r}.$$

Let X be a parameter matrix that has the same size as A and B and with a conforming partition into blocks:

$$X = \begin{bmatrix} X_{11} & \dots & X_{1t} \\ \vdots & \ddots & \vdots \\ X_{t1} & \dots & X_{tt} \end{bmatrix}, \qquad \text{e.g., size } X_{ii} = \text{size } H_{p_i}(\mu_i).$$

Block-wise solution

Let $(A, B) = (A_1, B_1) \oplus (A_2, B_2)$.

$$\begin{bmatrix} X_1^T & X_3^T \\ X_2^T & X_4^T \end{bmatrix} \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} + \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} = 0,$$
$$\begin{bmatrix} X_1^T & X_3^T \\ X_2^T & X_4^T \end{bmatrix} \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} + \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} = 0,$$

or equivalently

$$\begin{bmatrix} X_1^T A_1 + A_1 X_1 & X_3^T A_2 + A_1 X_2 \\ X_2^T A_1 + A_2 X_3 & X_4^T A_2 + A_2 X_4 \end{bmatrix} = 0,$$
$$\begin{bmatrix} X_1^T B_1 + B_1 X_1 & X_3^T B_2 + B_1 X_2 \\ X_2^T B_1 + B_2 X_3 & X_4^T B_2 + B_2 X_4 \end{bmatrix} = 0.$$

Block-wise solution for diagonal blocks in X

To determine X_1 consider the $(1,1)$-blocks:

$$\begin{bmatrix} X_1^T A_1 + A_1 X_1 & X_3^T A_2 + A_1 X_2 \\ X_2^T A_1 + A_2 X_3 & X_4^T A_2 + A_2 X_4 \end{bmatrix} = 0,$$
$$\begin{bmatrix} X_1^T B_1 + B_1 X_1 & X_3^T B_2 + B_1 X_2 \\ X_2^T B_1 + B_2 X_3 & X_4^T B_2 + B_2 X_4 \end{bmatrix} = 0,$$

giving the system

$$X_1^T A_1 + A_1 X_1 = 0,$$
$$X_1^T B_1 + B_1 X_1 = 0.$$

Block-wise solution for diagonal blocks in X

To determine X_1 consider the $(1,1)$-blocks:

$$\begin{bmatrix} X_1^T A_1 + A_1 X_1 & X_3^T A_2 + A_1 X_2 \\ X_2^T A_1 + A_2 X_3 & X_4^T A_2 + A_2 X_4 \end{bmatrix} = 0,$$
$$\begin{bmatrix} X_1^T B_1 + B_1 X_1 & X_3^T B_2 + B_1 X_2 \\ X_2^T B_1 + B_2 X_3 & X_4^T B_2 + B_2 X_4 \end{bmatrix} = 0,$$

giving the system

$$X_1^T A_1 + A_1 X_1 = 0,$$
$$X_1^T B_1 + B_1 X_1 = 0.$$

Three cases:

- $(A_1, B_1) : H_h(\mu), K_k,$ or M_m.

Block-wise solution for off-diagonal blocks of X

To determine X_2 and X_3 consider the corresponding off-diagonal blocks in the matrix equation:

$$\begin{bmatrix} X_1^T A_1 + A_1 X_1 & X_3^T A_2 + A_1 X_2 \\ X_2^T A_1 + A_2 X_3 & X_4^T A_2 + A_2 X_4 \end{bmatrix} = 0,$$
$$\begin{bmatrix} X_1^T B_1 + B_1 X_1 & X_3^T B_2 + B_1 X_2 \\ X_2^T B_1 + B_2 X_3 & X_4^T B_2 + B_2 X_4 \end{bmatrix} = 0,$$

since A_1, A_2, B_1, and B_2 are skew-symmetric we get the following system

$$X_3^T A_2 + A_1 X_2 = 0,$$
$$X_3^T B_2 + B_1 X_2 = 0.$$

Interaction between two canonical blocks

The following six cases occur:

- $(A_1, B_1) = H_k(\mu_1)$ $(A_2, B_2) = H_m(\mu_2)$;
- $(A_1, B_1) = K_k$ $(A_2, B_2) = K_m$;
- $(A_1, B_1) = M_k$ $(A_2, B_2) = M_m$;
- $(A_1, B_1) = H_k(\mu)$ $(A_2, B_2) = K_m$;
- $(A_1, B_1) = H_k(\mu)$ $(A_2, B_2) = M_m$;
- $(A_1, B_1) = K_k$ $(A_2, B_2) = M_m$.

Dimension of the solution space

Corollary

If the system

$$X^T A + AX = 0,$$
$$X^T B + BX = 0,$$

with (A, B) in canonical form, then the dimension of its solution space is equal to the sum

$$d_{(A,B)} = d_H + d_K + d_M + d_{HH} + d_{KK} + d_{MM} + d_{HK} + d_{HM} + d_{KM}$$

whose summands correspond to

- *the direct summands of (A, B) (diagonal blocks of X):*

$$d_H := 3\sum_{i=1}^{a} h_i, \qquad d_K := 3\sum_{i=1}^{b} k_i, \qquad d_M := c + 2\sum_{i=1}^{c} m_i;$$

Dimension of the solution space

Corollary

- *the pairs of direct summands of (A, B) of the* **same type** *(off-diagonal blocks of X):*

$$d_{HH} := 4 \sum_{\substack{i<j \\ \mu_i=\mu_j}} \min(h_i, h_j), \qquad d_{KK} := 4 \sum_{i<j} \min(k_i, k_j),$$

$$d_{MM} := \sum_{j<i} \left(2 \max(m_i, m_j) + \varepsilon_{ij}\right), \qquad \textit{where } \varepsilon_{ij} := \begin{cases} 2 & \textit{if } m_i = m_j, \\ 1 & \textit{if } m_i \neq m_j; \end{cases}$$

Dimension of the solution space

Corollary

- *the pairs of direct summands of* (A, B) *of the* **same type** *(off-diagonal blocks of X):*

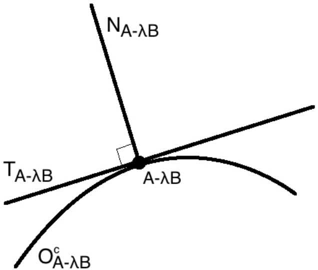
$$d_{HH} := 4 \sum_{\substack{i<j \\ \mu_i=\mu_j}} \min(h_i, h_j), \qquad d_{KK} := 4 \sum_{i<j} \min(k_i, k_j),$$

$$d_{MM} := \sum_{j<i} \left(2 \max(m_i, m_j) + \varepsilon_{ij}\right), \qquad \textit{where } \varepsilon_{ij} := \begin{cases} 2 & \textit{if } m_i = m_j, \\ 1 & \textit{if } m_i \neq m_j; \end{cases}$$

- *the pairs of direct summands of* (A, B) *of* **different types** *(off-diagonal blocks of X):*

$$d_{HK} := 0, \qquad d_{HM} := 2c \sum_i h_i, \qquad d_{KM} := 2c \sum_i k_i.$$

Tangent and normal subspaces



$$\mathcal{O}_C(A-\lambda B) = \{S^T(A-\lambda B)S \mid \det S \neq 0\}$$

$$\dim \mathcal{O}_C(A-\lambda B) := \dim \mathrm{T}_{A-\lambda B}$$

$$\operatorname{codim} \mathcal{O}_C(A-\lambda B) := \dim \mathrm{N}_{A-\lambda B}$$

$$\dim \mathcal{O}_C(A-\lambda B) + \operatorname{codim} \mathcal{O}_C(A-\lambda B) = n^2 - n$$

The vector space

$$\mathrm{T}_{A-\lambda B} \equiv \{(X^TA + AX) - \lambda(X^TB + BX) : X \in \mathbb{C}^{n\times n}\}$$

is the tangent space to the congruence orbit of $A-\lambda B$ at the point $A-\lambda B$. The orthogonal complement to $\mathrm{T}_{A-\lambda B}$ is called the normal space, $\mathrm{N}_{A-\lambda B}$, to the congruence orbit.

Codimension computations

Theorem

The codimension of the congruence orbit of $(A, B) \in \mathbb{C}_c^{n\times n} \times \mathbb{C}_c^{n\times n}$*, where* $\mathbb{C}_c^{n\times n}$ *is the space of skew-symmetric* $n \times n$ *matrices, can be calculated as follows:*

$$\operatorname{codim}(\mathcal{O}_C(A, B)) = d_{(A,B)} - n$$

where $d_{(A,B)} := \dim\left(\left\{X \in \mathbb{C}^{n\times n},\ \text{such that } \begin{matrix} X^T A + AX = 0 \\ X^T B + BX = 0 \end{matrix}\right\}\right)$.

Recall $d_{(A,B)} = d_H + d_K + d_M + d_{HH} + d_{KK} + d_{MM} + d_{HK} + d_{HM} + d_{KM}$.

A. Dmytryshyn, B. Kågström, V.V. Sergeichuk, Skew-symmetric matrix pencils: codimension counts and the solution of a pair of matrix equations, *Linear Algebra Appl.*, 438 (2013), pp. 3375–3396.

Symmetric matrix pencis

The same technique is applied in the case (A, B) are symmetric.

Theorem

The codimension of the congruence orbit of $(A, B) \in \mathbb{C}_s^{n\times n} \times \mathbb{C}_s^{n\times n}$, *where* $\mathbb{C}_s^{n\times n}$ *is the space of symmetric* $n \times n$ *matrices, can be calculated as follows:*

$$\operatorname{codim}(\mathcal{O}_C(A, B)) = d_{(A,B)} + n$$

where $d_{(A,B)} := \dim\left(\left\{X \in \mathbb{C}^{n\times n},\ \text{such that } \begin{array}{l} X^T A + AX = 0 \\ X^T B + BX = 0 \end{array}\right\}\right).$

A. Dmytryshyn, B. Kågström, V.V. Sergeichuk, Symmetric matrix pencils: Codimension counts and the solution of a pair of matrix equations, *Electron. J. Linear Algebra*, 27 (2014), pp. 1–18.

Matrix Canonical Structure Toolbox (in MATLAB)

For a skew-symmetric matrix pencil the following functions are implemented:

- construction of a canonical pair with specified canonical structure;
- construction of a pair (random) with specified canonical structure;
- computation of the codimension of the congruence orbit of a pencil.

A. Dmytryshyn, S. Johansson, B. Kågström, Codimension computations of congruence orbits of matrices, skew-symmetric and symmetric matrix pencils using Matlab, Report UMINF 13.18, 2013.

Matrix Canonical Structure Toolbox (in MATLAB)

For a skew-symmetric matrix pencil the following functions are implemented:

- construction of a canonical pair with specified canonical structure;
- construction of a pair (random) with specified canonical structure;
- computation of the codimension of the congruence orbit of a pencil.

A. Dmytryshyn, S. Johansson, B. Kågström, Codimension computations of congruence orbits of matrices, skew-symmetric and symmetric matrix pencils using Matlab, Report UMINF 13.18, 2013.

MCS Toolbox also works with:

- matrices up to similarity, congruence, and *congruence;
- matrix pencils (equivalence transformations);
- symmetric matrix pencils (congruence transformations);
- controllability and observability matrix pairs;
- state-space matrix pencils.

P. Johansson, Matrix canonical structure toolbox, Report UMINF 06.15, 2006.

Small perturbations of canonical forms

$$\left[\begin{array}{ccc|cc} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \lambda & \varepsilon & \delta \\ \hline & & & \lambda & 1 \\ & & & & \lambda \end{array}\right] \sim \begin{cases} \begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \lambda & 1 & \\ & & & \lambda & 1 \\ & & & & \lambda \end{bmatrix} & \text{if } \varepsilon \neq 0,\ \delta = 0, \\ \left[\begin{array}{cccc|c} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \lambda & 1 & \\ & & & \lambda & \\ \hline & & & & \lambda \end{array}\right] & \text{if } \varepsilon = 0,\ \delta \neq 0, \\ \left[\begin{array}{ccc|cc} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \lambda & & \\ \hline & & & \lambda & 1 \\ & & & & \lambda \end{array}\right] & \text{if } \varepsilon = 0,\ \delta = 0. \end{cases}$$

Jordan Canonical Form (JCF) is a discontinuous function of the entries!

Changes of JCF under small perturbations

$$\begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \lambda & 1 & \\ & & & \lambda & 1 \\ & & & & \lambda \end{bmatrix} \qquad \mathcal{O}_S(A) = \{ S^{-1}AS \mid \det S \neq 0 \}$$

$$\uparrow$$

$$\left[\begin{array}{cccc|c} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \lambda & 1 & \\ & & & \lambda & \\ \hline & & & & \lambda \end{array}\right] \qquad \left[\begin{array}{cccc|c} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \lambda & 1 & \\ & & & \lambda & \delta_1 \\ \hline & & & & \lambda \end{array}\right] \sim \begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \lambda & 1 & \\ & & & \lambda & 1 \\ & & & & \lambda \end{bmatrix}$$

$$\uparrow$$

$$\left[\begin{array}{ccc|cc} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \lambda & & \\ \hline & & & \lambda & 1 \\ & & & & \lambda \end{array}\right] \qquad \left[\begin{array}{ccc|cc} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \lambda & & \delta \\ \hline & & & \lambda & 1 \\ & & & & \lambda \end{array}\right] \sim \left[\begin{array}{cccc|c} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \lambda & 1 & \\ & & & \lambda & \\ \hline & & & & \lambda \end{array}\right]$$

Skew-symmetric matrix pencils or matrix pairs (A,B)

Let (A, B) be a pair of skew-symmetric complex matrices.

Congruence transformation (preserves skew-symmetry):

$$(A, B) \mapsto S^T(A, B)S := (S^TAS, S^TBS), \qquad \det S \neq 0.$$

$$\mathcal{O}_C(A, B) = \{S^T(A, B)S \mid \det S \neq 0\}.$$

Skew-symmetric matrix pencils or matrix pairs (A,B)

Let (A, B) be a pair of skew-symmetric complex matrices.

Congruence transformation (preserves skew-symmetry):

$$(A, B) \mapsto S^T(A, B)S := (S^T AS, S^T BS), \qquad \det S \neq 0.$$

$$\mathcal{O}_C(A, B) = \{S^T(A, B)S \mid \det S \neq 0\}.$$

Equivalence transformation DOES NOT!

$$(A, B) \mapsto P^{-1}(A, B)Q := (P^{-1}AQ, P^{-1}BQ), \qquad \det P \cdot \det Q \neq 0.$$

$$\mathcal{O}_E(A, B) = \{P^{-1}(A, B)Q \mid \det P \cdot \det Q \neq 0\}.$$

"Equivalence" ⇔ "Congruence"

Theorem

Let (A, B) and (C, D) be two pairs of skew-symmetric complex matrices. There exists a sequence of nonsingular matrices $\{R_n, S_n\}$ such that

$$R_n(C, D)S_n \to (A, B)$$

if and only if *there exists a sequence of nonsingular matrices $\{W_n\}$ such that*

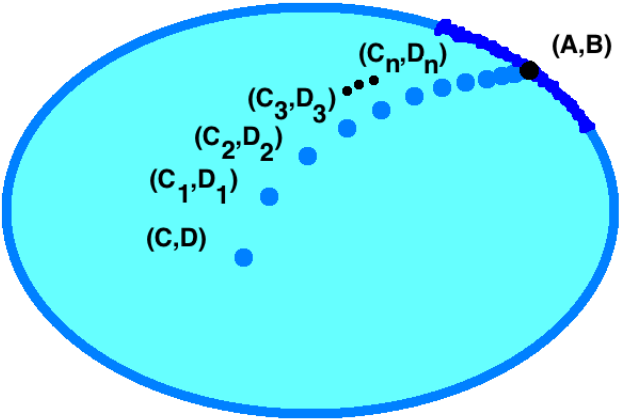
$$W_n^T(C, D)W_n \to (A, B).$$

Remark: the sufficiency is obvious

$W_n^T(C, D)W_n \to (A, B)$ implies $R_n(C, D)S_n \to (A, B)$, where $R_n := W_n^T, S_n := W_n$.

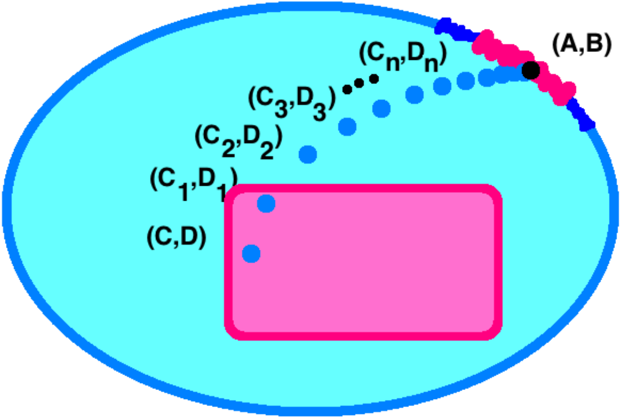
"Equivalence" ⇔ "Congruence"

There exist a sequence of pairs from the equivalence orbit of (C, D) $(C_1, D_1), (C_2, D_2), (C_3, D_3), \ldots$ such that $\|(C_n, D_n) - (A, B)\| \to 0$



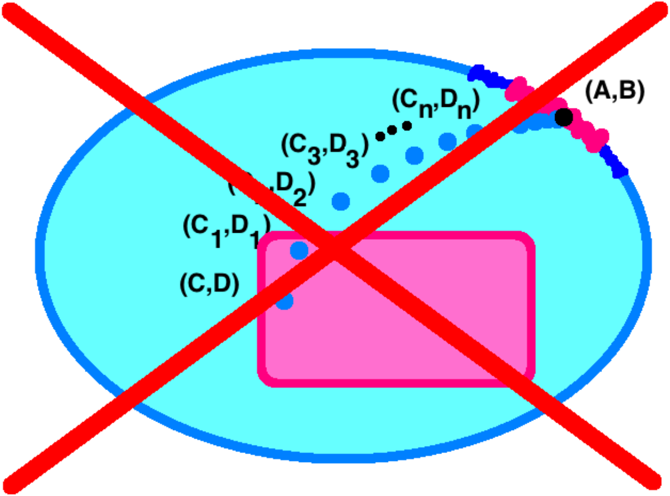
"Equivalence" ⇔ "Congruence"

What about the congruence orbits of (C, D) and (A, B)?



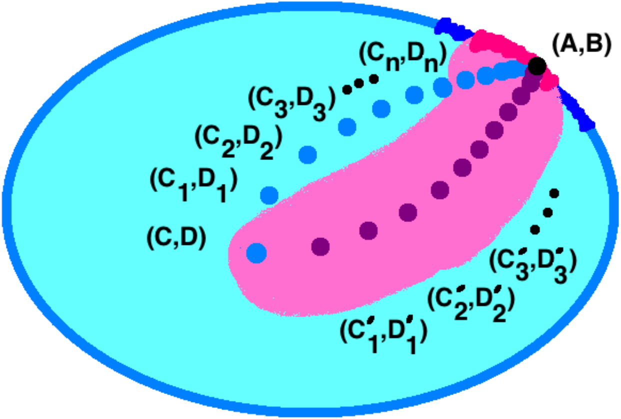
"Equivalence" ⇔ "Congruence"

What about the congruence orbits of (C, D) and (A, B)?



"Equivalence" ⇔ "Congruence"

There exist a sequence of (skew-symmetric) pairs from the congruence orbit of (C, D) $(C_1', D_1'), (C_2', D_2'), (C_3', D_3'), \ldots$ such that $\|(C_n', D_n') - (A, B)\| \to 0$



"Equivalence" ⇔ "Congruence"

Reformulation of the Theorem

If (A, B) and (C, D) are two pairs of skew-symmetric matrices then

$$\overline{\mathcal{O}_E}(C, D) \supset \mathcal{O}_E(A, B), \quad (\dim(\mathcal{O}_E(X, Y)) + \operatorname{codim}(\mathcal{O}_E(X, Y)) = 2n^2)$$
$$\Updownarrow$$
$$\overline{\mathcal{O}_C}(C, D) \supset \mathcal{O}_C(A, B), \quad (\dim(\mathcal{O}_C(X, Y)) + \operatorname{codim}(\mathcal{O}_C(X, Y)) = n(n-1))$$

"Equivalence" ⇔ "Congruence"

Reformulation of the Theorem

If (A, B) and (C, D) are two pairs of skew-symmetric matrices then

$$\overline{\mathcal{O}_E}(C, D) \supset \mathcal{O}_E(A, B), \quad (\dim(\mathcal{O}_E(X, Y)) + \operatorname{codim}(\mathcal{O}_E(X, Y)) = 2n^2)$$
$$\Updownarrow$$
$$\overline{\mathcal{O}_C}(C, D) \supset \mathcal{O}_C(A, B), \quad (\dim(\mathcal{O}_C(X, Y)) + \operatorname{codim}(\mathcal{O}_C(X, Y)) = n(n-1))$$

Proof idea: Assumption: $\overline{\mathcal{O}_E}(C, D) \supset \mathcal{O}_E(A, B)$.
Derive row and column permutations of (A, B) and (C, D):

$$(A, B) \sim \left(\begin{bmatrix} 0 & A_1 \\ -A_1^T & 0 \end{bmatrix}, \begin{bmatrix} 0 & B_1 \\ -B_1^T & 0 \end{bmatrix} \right) \text{ and } (C, D) \sim \left(\begin{bmatrix} 0 & C_1 \\ -C_1^T & 0 \end{bmatrix}, \begin{bmatrix} 0 & D_1 \\ -D_1^T & 0 \end{bmatrix} \right).$$

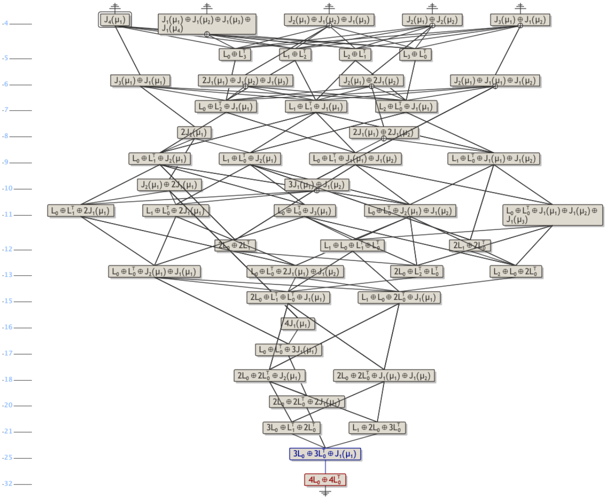
Use the closure relations for the equivalence orbits to prove:

$$\overline{\mathcal{O}_E}(C_1, D_1) \supset \mathcal{O}_E(A_1, B_1).$$

Conclusion: $\overline{\mathcal{O}_C}(C, D) \supset \mathcal{O}_C(A, B)$.

Stratification algorithm

Step 1: Stratification of 4×4 matrix pencils using StratiGraph.

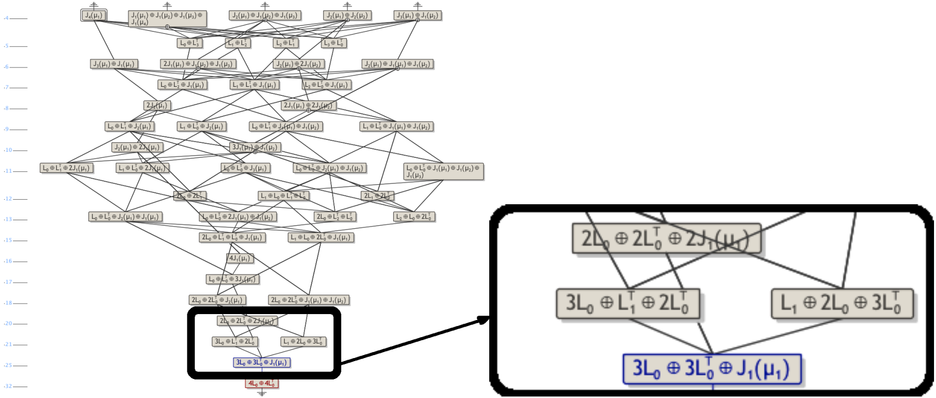


Nodes: orbits of the matrix pencils;

Path from (A, B) to (C, D): (A, B) can be transformed by a small perturbation to (C, D);

Stratification algorithm

Step 1: Stratification of 4×4 matrix pencils using StratiGraph.

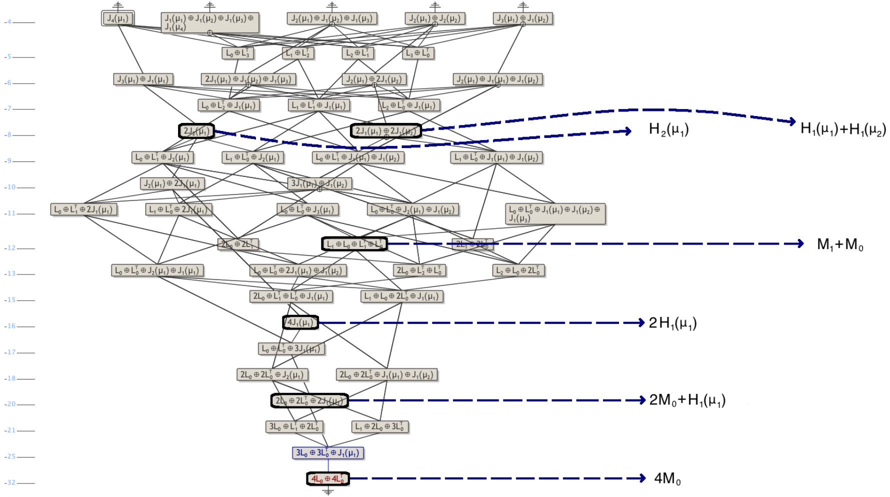


Nodes: orbits of the matrix pencils;

Path from (A, B) to (C, D): (A, B) can be transformed by a small perturbation to (C, D);

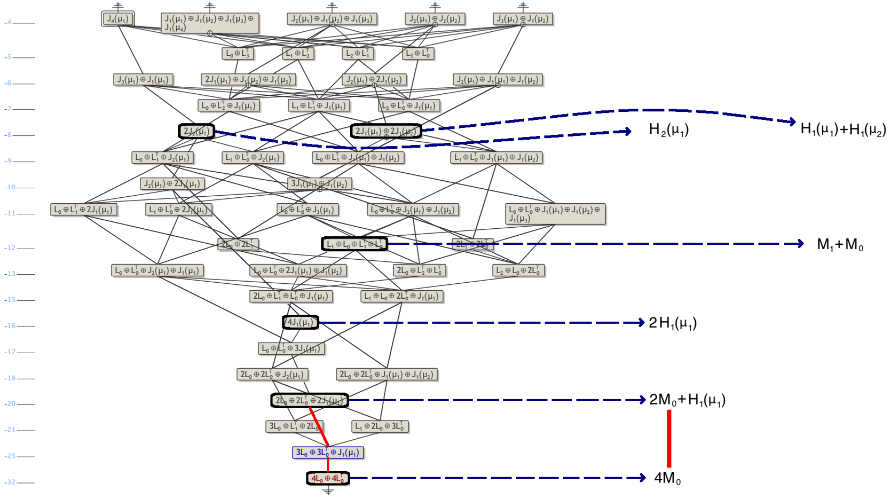
Stratification algorithm

Step 2: Extract all skew-symmetrized orbits.



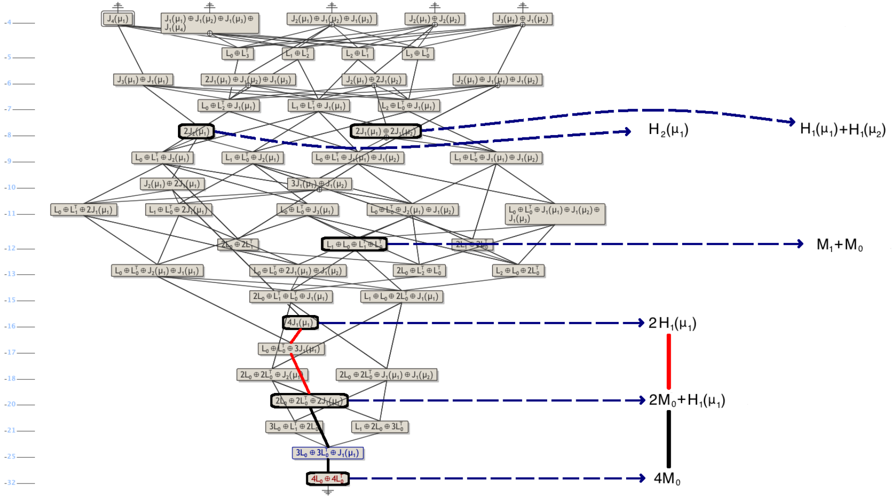
Stratification algorithm

Step 3: Examine the paths between the extracted orbits to obtain the closure hierarchy.



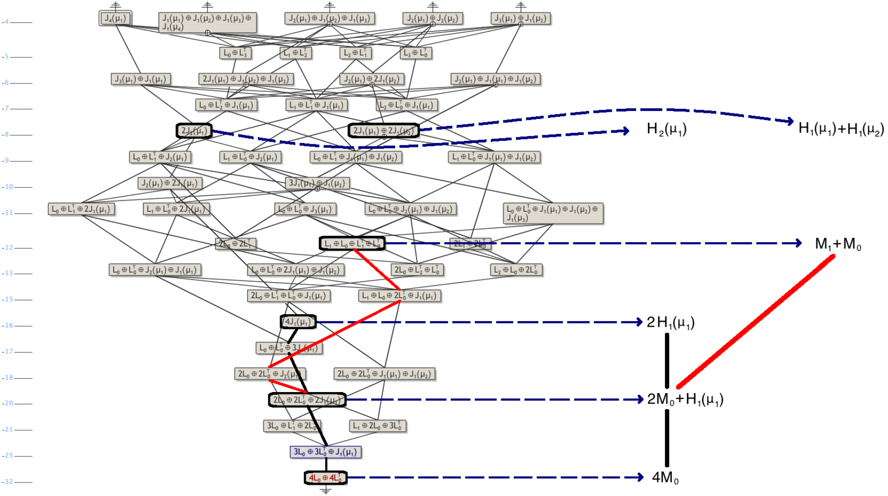
Stratification algorithm

Step 3: Examine the paths between the extracted orbits to obtain the closure hierarchy.



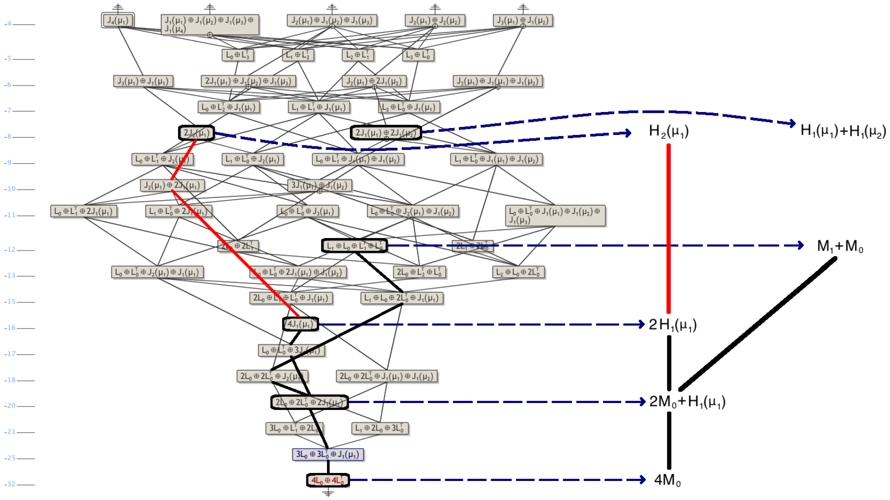
Stratification algorithm

Step 3: Examine the paths between the extracted orbits to obtain the closure hierarchy.



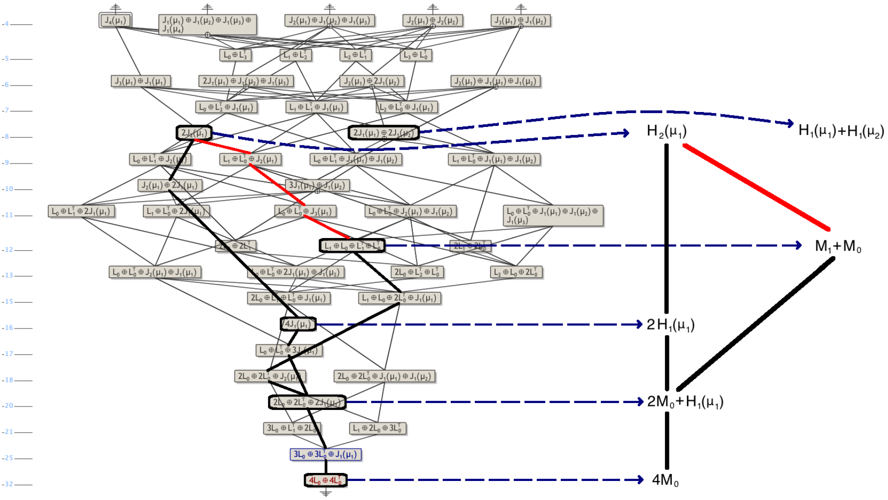
Stratification algorithm

Step 3: Examine the paths between the extracted orbits to obtain the closure hierarchy.



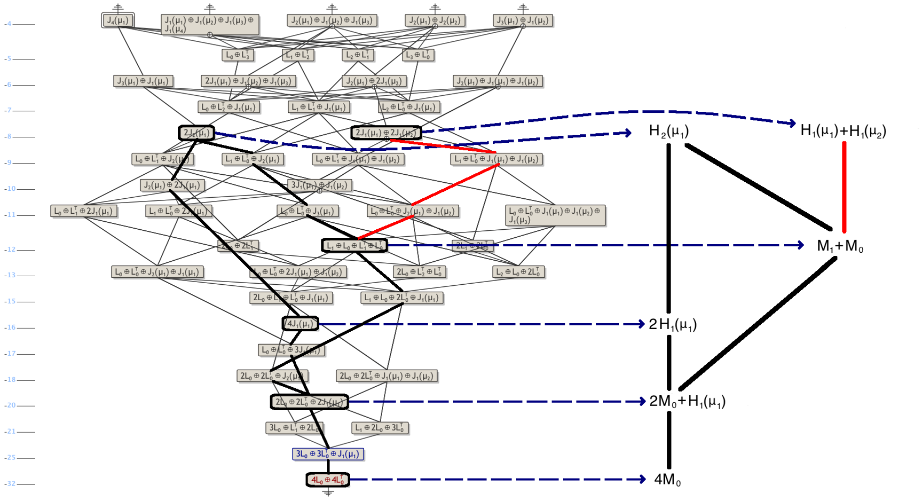
Stratification algorithm

Step 3: Examine the paths between the extracted orbits to obtain the closure hierarchy.



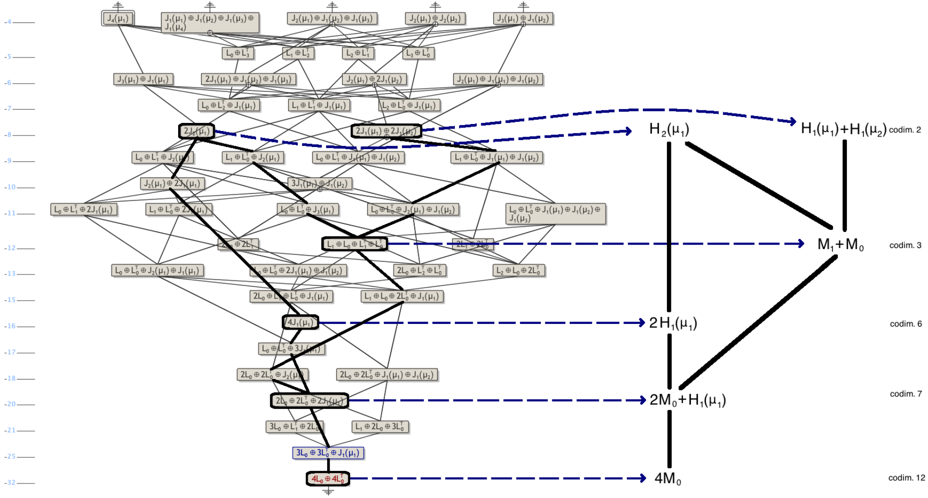
Stratification algorithm

Step 3: Examine the paths between the extracted orbits to obtain the closure hierarchy.



Stratification algorithm

Step 4: Compute and place the codimensions under congruence.



Congruence bundles

Definition

A bundle $\mathcal{B}_C(A - \lambda B)$ is a *union of skew-symmetric matrix pencil orbits* with the same singular structures and the same Jordan structures except that the distinct eigenvalues may be different.

Example

$H_5(2) \oplus H_3(2) \oplus H_4(5), H_5(0) \oplus H_3(0) \oplus H_4(11)$ and $H_5(9) \oplus H_3(9) \oplus H_4(1)$ are in the same bundle $H_5(\mu_1) \oplus H_3(\mu_1) \oplus H_4(\mu_2)$.

$$H_h(\mu) := \left(\begin{bmatrix} 0 & I_h \\ -I_h & 0 \end{bmatrix}, \begin{bmatrix} 0 & J_h(\mu) \\ -J_h(\mu)^T & 0 \end{bmatrix} \right), \quad \mu \in \mathbb{C},$$

Congruence bundles

Definition

A bundle $\mathcal{B}_C(A - \lambda B)$ is a *union of skew-symmetric matrix pencil orbits* with the same singular structures and the same Jordan structures except that the distinct eigenvalues may be different.

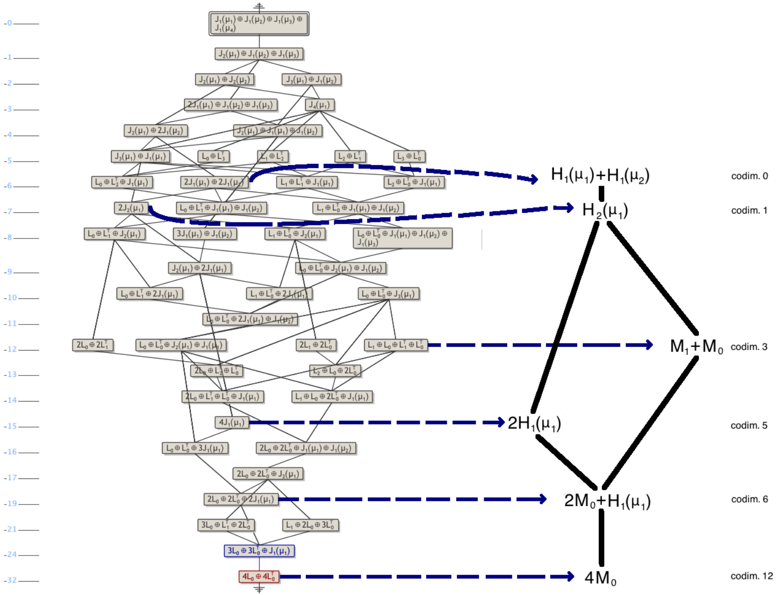
Example

$H_5(2) \oplus H_3(2) \oplus H_4(5)$, $H_5(0) \oplus H_3(0) \oplus H_4(11)$ and $H_5(9) \oplus H_3(9) \oplus H_4(1)$ are in the same bundle $H_5(\mu_1) \oplus H_3(\mu_1) \oplus H_4(\mu_2)$.

$$H_h(\mu) := \left(\begin{bmatrix} 0 & I_h \\ -I_h & 0 \end{bmatrix}, \begin{bmatrix} 0 & J_h(\mu) \\ -J_h(\mu)^T & 0 \end{bmatrix} \right), \quad \mu \in \mathbb{C},$$

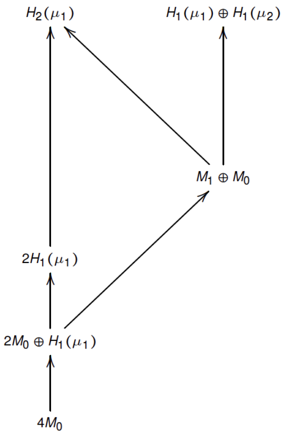
$$\operatorname{codim} \mathcal{B}_C(A - \lambda B) = \operatorname{codim} \mathcal{O}_C(A - \lambda B) - \# \{\text{distinct eigenvalues of } A - \lambda B\}$$

Stratification algorithm for bundles: 4×4 pencils

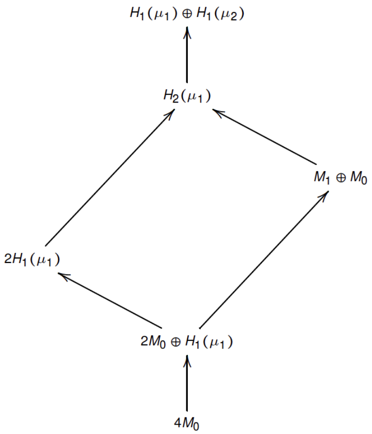


Stratification of 4 × 4 skew-symmetric pencils

Orbits



Bundles



A. Dmytryshyn and B. Kågström, Orbit closure hierarchies of skew-symmetric matrix pencils, Report UMINF 14.02 (submitted), 2014.

Symmetric/skew-symmetric matrix pencils

$A - \lambda B$, where $A = A^T$ and $B = -B^T$.

The codimentions are computed in

F. De Terán, F.M. Dopico, The solution of the equation $XA + AX^T = 0$ and its application to the theory of orbits, Linear Algebra Appl., 434 (2011), pp. 44–67.

The stratifications of 2×2 and 3×3 matrix pencils are computed in

A. Dmytryshyn, V. Futorny, B. Kågström, L. Klimenko, and V.V. Sergeichuk, Change of the congruence canonical form of 2-by-2 and 3-by-3 matrices under perturbations and bundles of matrices under congruence, Preprint, 2014.

Thank you!